Proposition (Extension of Fundamental Theorem of Calc.) Suppose that $w(t) = u(t) + i\,v(t)$ is continuous on $[a,b]$ and $W(t) = U(t) + i\,V(t)$ is differentiable such that $W'(t) = w(t)$ on $[a,b]$. Then

$$\int_a^b w(t)\,dt = W(b) - W(a).$$

Proof. Assume $w'(t) = w$. This means $U' = u$ and $V' = v$. Hence,

$$\begin{aligned}\int_a^b w(t)\,dt &= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \\ &= U(b) - U(a) + i\big(V(b) - V(a)\big) \quad (\text{by FTC}) \\ &= U(b) + iV(b) - \big(U(a) + iV(a)\big) \\ &= W(b) - W(a).\end{aligned}$$

This proves the claim.

Example We use the proposition to integrate e^{it} on $[0,\pi]$.

Notice that $\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{i}{i}e^{it} = e^{it}$. By the theorem

$$\begin{aligned}\int_0^{\pi} e^{it}\,dt = \left[\frac{1}{i}e^{it}\right]_0^{\pi} &= \frac{1}{i}e^{i\pi} - \frac{1}{i}e^{0} \\ &= \frac{1}{i}\left(e^{i\pi} - 1\right) \\ &= \frac{1}{i}\left(-1 - 1\right) = -\frac{2}{i} = 2i.\end{aligned}$$

//

Contours

So far, we have only defined the integral of a complex-valued function of a real variable over an interval. Integrals of complex-valued functions of a complex variable are defined on suitable curves in the complex plane, called contours.

Definition (arcs)

(1) An arc is a collection of points

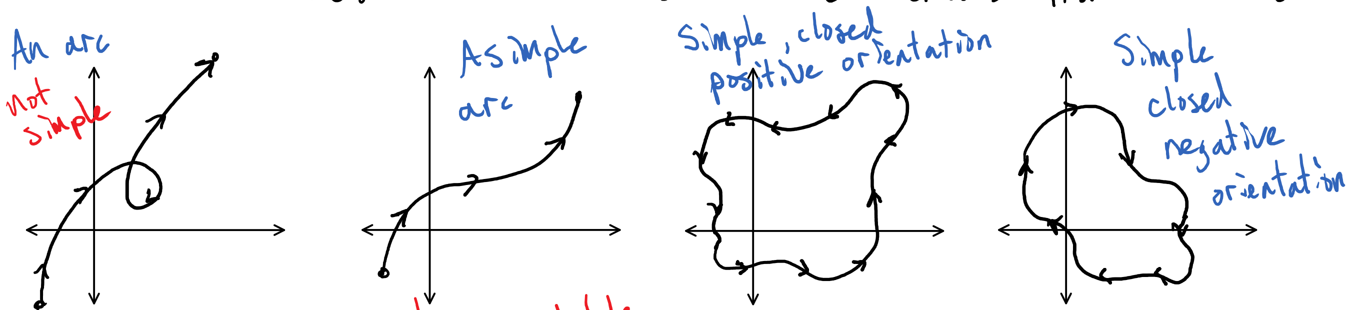
$$C = \{ z(t) : t \in [a,b] \}$$

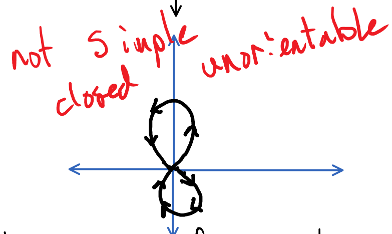
where $z(t) = x(t) + i y(t)$ and $x, y : [a,b] \longrightarrow \mathbb{R}$ are continuous functions. The function $z(t)$ is called a parameterization of C.

(2) An arc C is called simple or a Jordan arc if it does not cross itself: $z(t_1) = z(t_2) \Rightarrow t_1 = t_2$.

(3) If C is simple except for the fact that $z(a) = z(b)$, then C is called a simple closed curve or Jordan curve.

(4) A simple closed curve is positively oriented if it is traversed counter-clockwise as t increases from a to b.



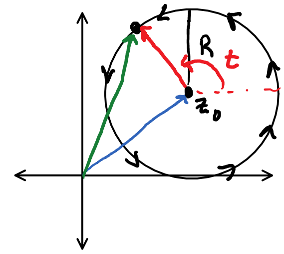


Example The most frequently encountered arcs and curves are line segments and circles.

(1) The circle of radius R centered at z_0 w/ positive orientation

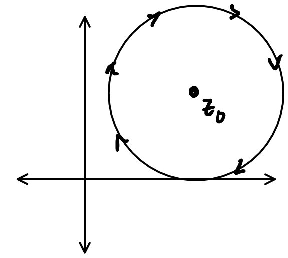
A parameterization is

$$z(t) = z_0 + R e^{it}, \quad t \in [0, 2\pi]$$



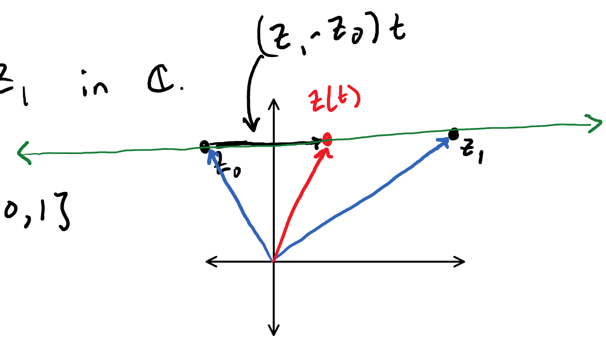
(2) The circle of radius R centered at z_0 w/ negative orientation

$$z(t) = z_0 + Re^{-it}, \quad t \in [0, 2\pi]$$



(3) The line segment from z_0 to z_1 in $\mathbb{C}$.

$$z(t) = z_0 + (z_1 - z_0)t, \quad t \in [0,1]$$



$\boxed{\text{Reparameterization of an arc}}$ Suppose that C is parameterized by

$$z(t) : [a,b] \longrightarrow \mathbb{C}.$$

A map

$$w(s) : [\alpha, \beta] \longrightarrow \mathbb{C}$$

is called an orientation-preserving reparameterization of C if there exists a surjective function

$$\phi : [\alpha, \beta] \longrightarrow [a,b]$$

with continuous derivative such that

w and z trace out same curve C

$$\underbrace{\phi(\alpha) = a}_{\text{preserves initial point}}, \quad \overbrace{\phi(\beta) = b}^{\text{preserves final pt}}, \quad \underbrace{\phi'(s) > 0}_{\text{strictly increasing}}, \text{ and } \overbrace{w(s) = z(\phi(t))}^{\text{w and z trace out same curve } C}. \quad //$$

$\boxed{\text{Definition (arc length / contours)}}$

(1) If C is parameterized by $z(t) = x(t) + iy(t)$ and $x'(t), y'(t)$ are continuous on $[a,b]$, then C is called a differentiable arc.

(2) The arc length of such a differentiable arc is

$$L = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

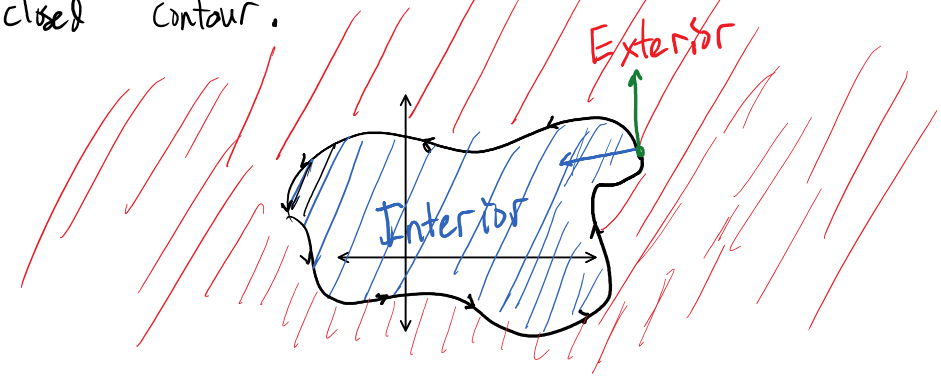
according to the definition from ordinary calculus.

(3) A differentiable arc C parameterized by $z(t)$ is called smooth if $z'(t) \neq 0$ on $[a,b]$.

(4) A contour is an arc consisting of a finite number of smooth arcs joined end to end. A simple closed contour is a contour that does not cross itself except that the initial and final points are the same. //

A deep theorem known as the Jordan Curve Theorem tells us that every simple closed contour C is the boundary of two distinct domains called the interior of C, which is bounded, and the exterior of C, which is unbounded.

The theorem is geometrically evident but the proof is not easy. We will assume it's truth so that we can refer to the interior of a simple closed contour.



Orientation can now be defined via right hand rule: point 4 fingers in the direction of the tangent vector, curl 4 fingers towards interior of curve. If your thumb points up, the orientation is positive. //

Contour Integration

Definition (contour integral) Suppose that $f: U \subseteq \mathbb{C} \to \mathbb{C}$ is a function and C is a contour lying in U. If C is parameterized by $z(t): [a,b] \to \mathbb{C}$ and $f(z(t))$ is piecewise continuous, then the *contour integral of f over C* is the integral

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

Note: since C is a contour, $z'(t)$ is piecewise continuous so that the integral exists.

Contour integrals are related to ordinary line integrals from calculus. Writing $f(z) = u(x,y) + i\,v(x,y)$ and $z(t) = x(t) + i\,y(t)$ we get:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt = \int_a^b \big(u(x(t),y(t)) + i\,v(x(t),y(t))\big)\big(x'(t) + i\,y'(t)\big)\, dt$$

$$= \int_a^b u(x(t),y(t))\,x'(t) - v(x(t),y(t))\,y'(t) \;\; dt \;\; i \int_a^b u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\, dt$$

$$= \int_a^b u\,dx - v\,dy \;+ i \int u\,dy + v\,dx$$

//

Proposition (Integral is Independent of parameterization) Suppose that $z: [a,b] \to \mathbb{C}$ parameterizes C and $w: [\alpha,\beta] \to \mathbb{C}$ is an orientation preserving reparameterization of C. Then

$$\int_C f(z)\,dz = \int_C f(w)\,dw.$$

Proof. Choose a function $\phi : [\alpha, \beta] \to [a,b]$ such that

$$\phi(\alpha) = a, \quad \phi(\beta) = b, \quad \phi'(s) > 0, \quad w(s) = z(\phi(s)).$$

Then

$$\begin{aligned} \int_C f(w)\,dw &= \int_\alpha^\beta f(w(t))\, w'(t)\,dt \\ &= \int_\alpha^\beta f(z(\phi(s)))\, z'(\phi(s)) \cdot \phi'(s)\,ds \\ &= \int_a^b f(z(t))\, z'(t)\,dt \\ &= \int_C f(z)\,dz. \end{aligned}$$

Set $t = \phi(s)$
$dt = \phi'(s)ds$
$\phi(\alpha) = a$
$\phi(\beta) = b$

$\blacksquare$

Notation (Contours)

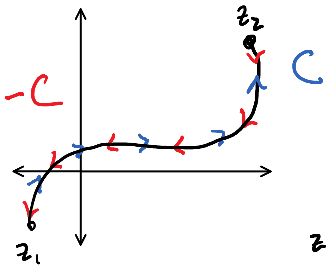
(1) Suppose C is a contour. Then $-C$ denotes the same set of points with opposite orientation. If $z(t) : [a,b] \to \mathbb{C}$ parameterizes C, then $w(t) = z(-t) : [-b,-a] \to \mathbb{C}$ parameterizes $-C$.

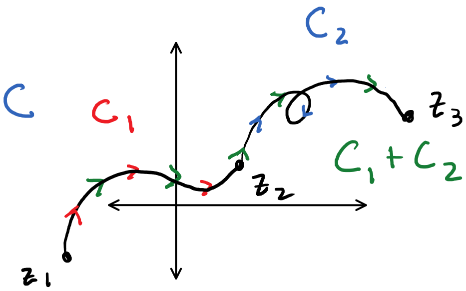
(2) If C_1 is a contour from z_1 to z_2 and C_2 from z_2 to z_3, then their sum is

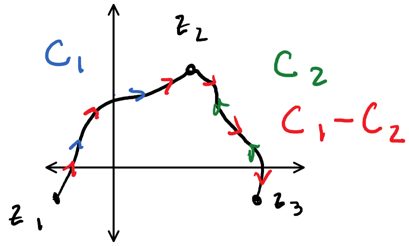
$$C = C_1 + C_2$$

is the contour obtained by traversing C_1 and then C_2. If C_1 and C_2 have the same final point, then the sum of C_1 and $-C_2$ is defined and is written

$$C_1 - C_2 = C_1 + (-C_2).$$







Proposition (Properties of Contour Integral) Assume f, g are piecewise continuous on an contour used.

(1) $\int_C z_0 f(z)\,dz = z_0 \int_C f(z)\,dz$, $z_0 \in \mathbb{C}$;

(2) $\int_C f(z) + g(z)\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$;

(3) $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

(4) $\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$ if $C = C_1 + C_2$.

Proof.

(1) $$\int_C z_0 f(z)\,dz = \int_a^b z_0 f(z(t))\, z'(t)\,dt$$

follows from previous results.

$$= z_0 \int_a^b f(z(t))\, z'(t)\,dt$$

$$= z_0 \int_C f(z)\,dz$$

(2) Follows from previous results.

(3) Suppose C is parameterized by $z(t) : [a,b] \to \mathbb{C}$. Then a parameterization for $-C$ is $w(t) = z(-t) : [-b,-a] \to \mathbb{C}$.

$$\int_{-C} f(w)\,dw = \int_{-b}^{-a} f(w(t))\, w'(t)\,dt$$

$s = -t$
$ds = -dt$
$s(-a) = a$
$s(-b) = b$

$$= -\int_{-b}^{-a} f(z(-t))\, z'(-t)\, dt$$

$$= \int_{b}^{a} f(z(s))\, z'(s)\, ds = -\int_{a}^{b} f(z(s))\, z'(s)\, ds$$

from previous results.

$$= -\int_C f(z)\, dz.$$

(4) Exercise for the motivated student.

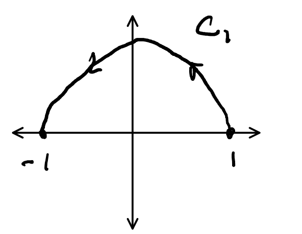
Examples of Contour Integration

(1) Integrate $f(z) = \frac{1}{z}$ over the following contours:

C_1: upper half of unit circle, from 1 to -1

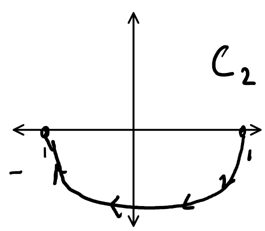
C_2: lower half of unit circle, from 1 to -1

C_3: $C_1 - C_2$



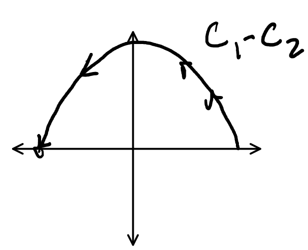
For C_1: parameterize C_1 as $z(t) = e^{it}$, $0 \le t \le \pi$.
Then

$$\int_{C_1} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{it}}\, ie^{it}\, dt = i\int_0^{\pi} 1\, dt = \pi i.$$



For C_2: parameterize C_2 as $z(t) = e^{-it}$, $0 \le t \le \pi$.
Then

$$\int_{C_2} \frac{1}{z}\, dz = \int_0^{\pi} \frac{1}{e^{-it}}\, -ie^{-it}\, dt = -i\int_0^{\pi} 1\, dt = -\pi i.$$



For C_3

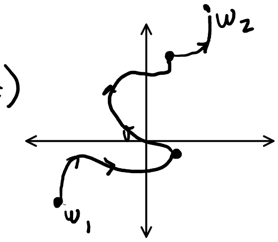
$$\int_{C_3} \frac{1}{z}\, dz = \int_{C_1 - C_2} \frac{1}{z}\, dz = \int_{C_1} \frac{1}{z}\, dz + \int_{-C_2} \frac{1}{z}\, dz$$

$$= \int_{C_1} \frac{1}{z}\, dz - \int_{C_2} \frac{1}{z}\, dz = \pi i - (-\pi i) = 2\pi i.$$

This example shows: the integral may depend on the path taken and not just the endpoints. Also, the integral over a closed contour may be nonzero.

(2) Integrate $f(z) = z$ over <u>any</u> contour C connecting a point w_1 to a point w_2.

First, suppose C is a smooth arc joining w_1 to w_2 and parameterized by $z : [a,b] \to \mathbb{C}$.



Since $\frac{d}{dt}\left(\frac{1}{2} z(t)^2\right) = \frac{1}{2}\left(z(t) z'(t) + z(t) z'(t)\right)$
$= z(t) z'(t)$.

$$\int_C z\, dz = \int_a^b z(t) z'(t)\, dt$$
$$= \frac{1}{2} z(b)^2 - \frac{1}{2} z(a)^2 = \frac{w_2^2 - w_1^2}{2}.$$

Now, if C is a contour, it can be written a sum of C_i, $i = 1, \ldots, n$

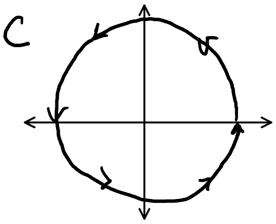
where C_i is a smooth arc joining z_i to z_{i+1}, $z_1 = w_1$, $z_{n+1} = w_2$.

Then
$$\int_C z dz = \sum_{i=1}^n \int_{C_i} z\, dz = \sum_{i=1}^n \frac{z_{i+1}^2 - z_i^2}{2}$$
$$= \frac{z_{n+1}^2 - z_1^2}{2}$$
$$= \frac{w_2^2 - w_1^2}{2}.$$

This example shows that some integrals depend only on the endpoints and not the path taken. Also, if $w_2 = w_1$, then we showed that
$$\int_C z dz = 0$$
for any closed contour C. //

(3) Integrate $f(z) = z^m \bar{z}^n$, $m, n \in \mathbb{Z}$, over the unit circle.

C



Parameterize C as $z(t) = e^{it}$, $0 \le t \le 2\pi$.

Then

$$\int_C z^m \bar{z}^n \, dz = \int_0^{2\pi} (e^{it})^m \left(\overline{e^{it}}\right)^n i e^{it} \, dt$$

$$= i \int_0^{2\pi} e^{imt} \left(e^{\overline{it}}\right)^n e^{it} \, dt$$

$$= i \int_0^{2\pi} e^{imt} \, e^{-int} \, e^{it} \, dt$$

$$= i \int_0^{2\pi} e^{i(m-n+1)t} \, dt$$

Case 1: $m = n-1$

$$= i \int_0^{2\pi} 1 \, dt = 2\pi i$$

Case 2: $m \neq n-1$

$$= i \left(\left[\frac{1}{i(m-n+1)} e^{i(m-n+1)t} \right]_0^{2\pi} \right)$$

$$= \frac{1}{m-n+1} \left(e^{i(m-n+1)\cdot 2\pi} - e^0 \right)$$

integer mult. of 2π

$$= \frac{1}{m-n+1} (1 - 1) = 0.$$